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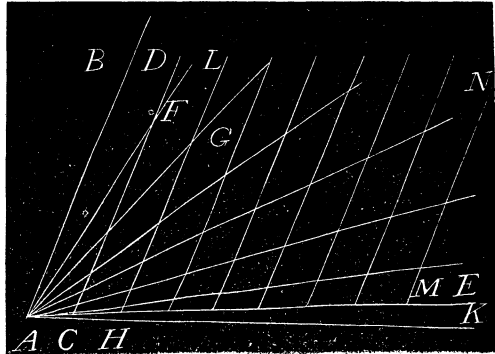
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furnish a second case of B , which is impossible. For instance, when it has been proved that the bisector of an angle of a triangle cuts the base in the ratio of the adjacent sides, the "opposite" and converse follow at once, because there is but one point in which a line can be cut in a given ratio. Now Euclid's assumption was the uniqueness of the notion B ; that is, "through a given point but one line can be drawn parallel to (i. e. so as not to meet) a given line." The demonstration in Crelle of this proposition is a direct demonstration of II above, I being supposed previously established; it is substantially as follows:—

Let AB and CD be line making equal angles with the secant AE , so that AB and CL cannot meet, and let AF be any other line in the plane passing through A ; then AF and CD will meet on that side of AE , on which the interior angles are less than two right angles.

From A draw a series of lines cutting off from FAE successive angles equal BAF , until we come to a line AK falling on the other side of AE , and let BAK contain n angles equal to BAF . Then, as the successive angles may be shown equal by coincidence, the space BAK (the lines being produced indefinitely) is n times the space BAF . Also draw a series of lines making the same angles with AE that AB makes, and cutting off from AE successive distances equal to AC , and let AM contain n distances equal to AC . Then the spaces $BACD$, $DCHL$, &c. may be shown equal by coincidence, therefore the space $BAMN$ equals n times the space $BACD$. Now the space BAK is greater than the space $BAMN$ therefore the space BAF is greater than $BACD$. Now if AF did not meet CD the space BAF would be less than the space $BACD$. Hence AF will if sufficiently produced meet CD .



PERFECT CUBES.

BY PROF. D. M. SENSENIG, INDIANA, PA.

THE object of this article is to resolve the general cubic polynomial $ax^3 + bx^2 + cx + d$, into special forms that may be rendered perfect cubes for particular values of x .

Put $ax^3 + bx^2 + cx + d = (px + q)^3$,
then $(a - p^3)x^3 + (b - 3p^2q)x^2 + (c - 3pq^2)x + d - q^3 = 0$.

Sup. 1st. Put $q^3 = d = d_1^3$, or $q = d_1$,
then $ax^3 + bx^2 + cx + d = (px + d_1)^3; \dots\dots\dots (A)$
and $(a - p^3)x^3 + (b - 3p^2d_1)x + c - 3pd_1^2 = 0. \dots\dots\dots (X)$

Sup. 2nd. Put $p^3 = a = a_1^3$, or $p = a_1$,
then $a_1^3x^3 + bx^2 + cx + d = (a_1x + q)^3; \dots\dots\dots (B)$
and $(b - 3a_1^2q)x^2 + (c - 3a_1q^2)x + d - q^3 = 0. \dots\dots\dots (Y)$

Sup. 3rd. Put $3p^2q = b$, or $3p^2q_1^2 = 3b_1^2$, or $p = b_1 \div q_1$,
then $ax^3 + 3b_1^2x^2 + cx + d = \left(\frac{b_1}{q_1}x + q_1^2\right)^3; \dots\dots\dots (C)$
and $\left(a - \frac{b_1^3}{q_1^3}\right)x^3 + (c - 3b_1q_1^3)x + d - q_1^6 = 0. \dots\dots\dots (Z)$

Sup. 4th. Put $3pq^2 = c$, or $3p_1^2q^2 = 3c_1^2$, or $q = c_1 \div p_1$,
than $ax^3 + bx^2 + 3c_1^2x + d = \left(\frac{c_1}{p_1} + p_1^2x\right)^3; \dots\dots\dots (D)$
and $(a - p_1^6)x^3 + (b - 3p_1^3c)x^2 + d - c_1^3 \div p_1^3 = 0. \dots\dots\dots (W)$

I. Resuming equations X and A :

1st. Put $3pd_1^2 = c$, or $p = c \div 3d_1^2$, in (X) and reduce, then
 $x = \frac{9c^2d_1^3 - 27bd_1^6}{27ad_1^6 - c^3}$, and $ax^3 + bx^2 + cx + d = \left(\frac{2c^2d_1 - 9bd_1^4c + 27ad_1^7}{27ad_1^6 - c^3}\right)^3$.
 $\therefore x = \frac{9c^2d^3 - 27bd^6}{27ad^6 - c^3}$ renders $ax^3 + bx^2 + cx + d = \left(\frac{2c^3d - 9bd^4c + 27ad^7}{27ad^6 - c^3}\right)^3. (A')$

Cor. Put $2c^3d - 9bd^4c + 27ad^7 = 0$, or $b = (2c^3 + 27ad^6) \div 9cd^3$, in (A') and reduce, then

$$x = \frac{3c^2d^3 - 81ad^9}{27acd^6 - c^4} \text{ renders } ax^3 + \frac{2c^3 + 27ad^6}{9cd^3}x^2 + cx + d^3 = 0. \dots (a')$$

2nd. Put $p^3 = a = a_1^3$, or $p = a_1$ in (X) , and reduce, then

$$x = \frac{3a_1d_1^2 - c}{b - 3a_1^2d_1}, \text{ and } a_1^3x^3 + bx^2 + cx + d = \left(\frac{bd_1 - c}{b - 3a_1^2d_1}\right)^3.$$

$$\therefore x = \frac{3ad^2 - c}{b - 3a^2d} \text{ renders } a^3x^3 + bx^2 + cx + d = \left(\frac{bd - c}{b - 3a^2d}\right)^3. \dots (A'')$$

Cor. Put $bd - c = 0$, or $c = bd$, in (A'') , and reduce, then

$$x = \frac{3ad^2 - bd}{b - 3a^2d} \text{ renders } a^3x^3 + bx^2 + bdx + d^3 = 0. \dots (a'')$$

3rd. Put $3p^2d_1 = b$, or $3p^2 = 3b_1^2 \div d_2^2$, or $p = b_1 \div d_2$, in (X) and reduce, then

$$x^2 = \frac{3b_1d_2^6 - cd_2^3}{ad_2^3 - b_1^3} = \frac{d_2^2}{(ad_2^3 - b_1^3)^2} \left[(3b_1d_2^4 - cd_2)(ad_2^3 - b_1^3) \right];$$

whence $x = d_2 \sqrt{(3b_1d_2^4 - cd_2)(ad_2^3 - b_1^3)} \div (ad_2^3 - b_1^3)$.

Put $3b_1d_2^4 - cd_2 = n^2(ad_2^3 - b_1^3)$, or $c = [3b_1d_2^4 - n^2(ad_2^3 - b_1^3)] \div d_2$, then $x = nd_2$, and $ax^3 + 3b_1x^2 + [3b_1d_2^4 - n^2(ad_2^3 - b_1^3)]x \div d_2 + d_2^6$
 $= (b_1n + d_2^2)^3$.

$\therefore x = nd$ renders $ax^3 + 3b^2x^2 + [3bd^4 - n^2(ad^3 - b^3)]x \div d + d^6$
 $= (bn + d^2)^3 \dots (A''')$

Cor. Put $bn + d^2 = 0$, or $n = -d^2 \div b$, in (A''') and reduce, then $x = -d^3 \div b$ renders $ax^3 + 3b^2x^2 + (4b^3d^4 - ad^7)x \div b^2d + d^6 = 0 \dots (a''')$

II. Resuming equations (Y) and (B):

1st. Put $q^3 = d = d_1^3$, or $q = d_1$, in (Y) and reduce, then

$$x = \frac{3a_1d_1^2 - c}{b - 3a_1^2d_1}, \text{ and } a_1^3x^3 + bx^2 + cx + d_1^3 = \left(\frac{bd_1 - a_1c}{b - 3a_1^2d_1} \right)^3.$$

$\therefore x = \frac{3ad^2 - c}{b - 3a^2d}$ renders $a^3x^3 + bx^2 + cx + d^3 = \left(\frac{bd - ac}{b - 3a^2d} \right)^3 \dots (B')$

Cor. Put $bd = ac$, or $c = bd \div a$, in (B') and reduce, then

$$x = \frac{3a^2d^2 - bd}{ab - 3a^3d} \text{ renders } a^3x^3 + bx^2 + \frac{bd}{a}x + d^3 = 0 \dots (b')$$

2nd. Put $3a_1^2q = b$, or $q = b \div 3a_1^2$, in (Y) and reduce, then

$$x = \frac{b^3 - 27a_1^6d}{27a_1^6c - 9a_1^3b^2}, \text{ and } a_1^3x^3 + bx^2 + cx + d = \left(\frac{9a_1^3bc - 2b^3 - 27a_1^6d}{27a_1^6c - 9a_1^3b^2} \right)^3.$$

$\therefore x = \frac{b^3 - 27a^6d}{27a^6c - 9a^3b^2}$ renders $a^3x^3 + bx^2 + cx + d = \left(\frac{9a^3bc - 2b^3 - 27a^6d}{27a^6c - 9a^3b^2} \right)^3$
 $\dots (B'')$

Cor. Put $9a^3bc - 2b^3 - 27a^6d = 0$, or $d = (9a^3bc - 2b^3) \div 27a^6$, in (B'') and reduce, then

$$x = \frac{b^3 - 3a^3bc}{9a^6c - 3a^3b^2} \text{ renders } a^3x^3 + bx^2 + cx + \frac{9a^3bc - 2b^3}{27a^6} = 0 \dots (b'')$$

3rd. Put $3a_1q^2 = c$, or $3a_2^2q^2 = 3c_1^2$, or $q = c_1 \div a_2$, in (Y) and reduce, then

$$x^2 = \frac{c_1^3 - a_2^3d}{a_2^3b - 3a_2^4c_1} = \frac{1}{a_2^2(a_2b - 3a_2^4c_1)} \left[(c_1^3 - a_2^3d)(a_2b - 3a_2^4c_1) \right];$$

whence $x = \frac{1}{a_2(a_2b - 3a_2^4c_1)} \sqrt{[(c_1^3 - a_2^3d)(a_2b - 3a_2^4c_1)]}$.

Put $c_1^3 - a_2^3 d = n^2(a_2 b - 3a_2^4 c_1)$, or $d = [c_1^3 - n^2(a_2 b - 3a_2^4 c_1)] \div a_2^3$,
 then $x = \frac{n}{a_2}$, and $a_2^6 x^3 + b x^2 + 3c_1^2 x + \frac{c_1^3 - n^2(a_2 b - 3a_2^4 c_1)}{a_2^3} = \left(\frac{a_2^2 n + c_1}{a}\right)^3$.
 $\therefore x = \frac{n}{a}$ renders $a^6 x^3 + b x^2 + 3c^2 x + \frac{c^3 - n^2(ab - 3a^4 c)}{a^3} = \left(\frac{a^2 n + c}{a}\right)^3. (B''')$

Cor. Put $a^2 n + c = 0$, or $n = -c \div a^2$, in (B''') and reduce, then

$$x = -\frac{c}{a^3} \text{ renders } a^6 x^3 + b x^2 + 3c x + \frac{c^3 - abc + 3a^4 c^2}{a^7} = 0 \dots (b''')$$

III. Resuming equations (Z) and (C) :

1st. Put $b_1^3 = a q_1^3 = a_1^3 q_1^3$, or $q_1 = b_1 \div a_1$, in (Z) and reduce, then
 $x = \frac{b_1^6 - a_1^6 d}{a_1^6 c - 3a_1^3 b_1^4}$ and $a_1^3 x^3 + 3b_1^2 x^2 + c x + d = \left(\frac{a_1^3 b_1 c - 2b^6 - a_1^6 d}{a_1^5 c - 3a_1^2 b_1^4}\right)^3$.
 $\therefore x = \frac{b^6 - a^6 d}{a^6 c - 3a^3 b^4}$ renders $a^3 x^3 + 3b^2 x^2 + c x + d = \left(\frac{a^3 b c - 2b^6 - a^6 d}{a^5 c - 3a^2 b^4}\right)^3. (C')$

Cor. Put $a^3 b c - 2b^6 - a^6 d = 0$, or $d = (a^3 b c - 2b^6) \div a^6$, in (C') and reduce, then

$$x = \frac{3b^6 - a^3 b c}{a^6 c - 3a^3 b^4} \text{ renders } a^3 x^3 + 3b^2 x^2 + c x + \frac{a^3 b c - 2b^6}{a^6} = 0 \dots (c')$$

2nd. Putting $q_1 = d = d_1^6$, or $q_1 = d_1$, in (Z) and (C) produces (A''') and hence will be omitted.

3rd. Put $3b_1 q_1^3 = c$, or $3b_2^3 q_1^3 = 3c_1^3$, or $q_1 = c_1 \div b_2$, in (Z) and reduce, then

$$x^3 = \frac{c_1^3 - b_2^6 c_1^3 d}{a b_2^6 c^3 - b_2^{12}} = \frac{c_1^3}{b_2^6 (a c_2^3 - b_2^{12})^3} \left[(c_1^6 - b_2^6 d) (a c_1^3 - b_2^{12})^2 \right];$$

$$\text{whence } x = \frac{c_1}{b_2^2 (a c_1^3 - b_2^{12})} \left[(c_2^6 - b_2^6 d) (a c_1^3 - b_2^{12})^2 \right]^{\frac{1}{3}}.$$

Put $c_1^6 - b_2^6 d = n^3(a c_1^3 - b_2^{12})$, or $d = [c_1^6 - n^3(a c_1^3 - b_2^{12})] \div b_2^6$; then

$$x = \frac{c_1 n}{b_2^2}, \text{ and } a x^3 + 3b_2^6 x^2 + 3c_1^2 x + \frac{c_1^6 - n^3(a c_1^3 - b_2^{12})}{b_2^6} = \left(\frac{b_2^4 n + c^2}{b_2^2}\right)^3.$$

$$\therefore x = \frac{c n}{b^2} \text{ renders } a x^3 + 3b^6 x^2 + 3c^3 x + \frac{c^6 - n^3(a c^3 - b^{12})}{b^6} = \left(\frac{b^4 n + c^2}{b^2}\right)^3.$$

..... (C'')

Cor. Put $b^4 n + c^2 = 0$, or $n = -c^2 \div b^4$, in (C'') and reduce, then

$$x = -c^3 \div b^6 \text{ renders } a x^3 + 3b^6 x^2 + 3c^3 x + a c^9 \div b^{18} = 0 \dots (c'')$$

IV. Resuming equations (W) and (D) :

1st. Put $d = c_1^3 \div p_2^3$, or $p_1^3 d_1^3 = c_1^3$, or $p_1 = c_1 \div d_1$, in (W) and reduce, then

$$x = \frac{3c_1^4 d_1^3 - b d_1^6}{a d_1^6 - c_1^6} \text{ and } ax^3 + bx^2 + 3c_1 x + d_1^3 = \left(\frac{2c_1^6 d_1 - b c_1^2 d_1^4 + a d_1^7}{a d_1^6 - c_1^6} \right)^3.$$

$$\therefore x = \frac{3c^4 d^3 - b d^6}{a d^6 - c^6} \text{ renders } ax^3 + bx^2 + 3cx + d^3 = \left(\frac{2c^6 d - b c^2 d^4 + a d^7}{a d^6 - c^6} \right)^3. \quad \dots (D')$$

Cor. Put $2c^6 d - b c^2 d^4 + a d^7 = 0$, or $b = (2c^6 d + a d^7) \div c^2 d^4$, in (D') and reduce, then

$$x = \frac{c^6 d^3 - a d^9}{a c^2 d^6 - c^8} \text{ renders } ax^3 + \frac{2c^6 d + a d^7}{c^2 d^4} x^2 + 3c^2 x + d^3 = 0 \dots (d')$$

2nd. Put $p_1^6 = a = a_1^6$, or $p_1 = a_1$, in (W) and reduce, then

$$x^2 = \frac{c_1^3 - a_1^3 d}{a_1^3 b - 3a_1^6 c_1} = \frac{1}{(a_1^3 b - 3a_1^6 c_1)^2} \left[(c_1^3 - a_1^3 d)(a_1^3 b - 3a_1^6 c_1) \right];$$

$$\text{whence } x = \frac{1}{a_1^3 b - 3a_1^6 c_1} \sqrt{[(c_1^3 - a_1^3 d)(a_1^3 b - 3a_1^6 c_1)]}.$$

$$\text{Put } c_1^3 - a_1^3 d = n^2(a^3 b - 3a_1^6 c_1), \text{ or } d = [c_1^3 - n^2(a_1^3 b - 3a_1^6 c_1)] \div a_1^3, \\ \text{then } x = n, \text{ and } a_1^6 x^3 + b x^2 + 3c_1^2 x + \frac{c_1^3 - n^2(a_1^3 b - 3a_1^6 c_1)}{a_1^3} = \left(\frac{a_1^3 n + c_1}{a_1} \right)^3.$$

$$\therefore x = n \text{ renders } a^6 x^3 + b x^2 + 3c^2 x + \frac{c^3 - n^2(a^3 b - 3a^6 c)}{a^3} = \left(\frac{a^3 n + c}{a} \right)^3. \quad \dots (D'')$$

Cor. Put $a^3 n + c = 0$, or $n = -c \div a^3$, in (D'') and reduce, then $x = -c \div a^3$ renders $a^6 x^3 + b x^2 + 3c^2 x + (4a^3 c^3 - b c^2) \div a^6 = 0$.

$\dots (d'')$

3rd. Put $3p_1^3 c_1 = b = 3b_1^3$, or $p_1^3 = b_1^3 \div c_2^3$, or $p_1 = b_1 \div c_2$, in (W) and reduce, then

$$x^3 = \frac{c_2^6 (c_2^{12} - b_1^3 d)}{b_1^3 (a c_2^6 - b_1^6)} = \frac{c_2^6}{b_1^3 (a c_2^6 - b_1^6)^3} \left[(c_2^{12} - b_1^3 d)(a c_2^6 - b_1^6)^2 \right];$$

$$\text{whence } x = \frac{c_2^2}{b_1 (a c_2^6 - b_1^6)} \left[(c_2^{12} - b_1^3 d)(a c_2^6 - b_1^6)^2 \right]^{\frac{1}{3}}.$$

Put $c_2^{12} - b_1^3 d = n^3(a c_2^6 - b_1^6)$, or $d = [c_2^{12} - n^3(a c_2^6 - b_1^6)] \div b_1^3$, then

$$x = \frac{c_2^2 n}{b_1}, \text{ and } ax^3 + 3b_1^3 x^2 + 3c_2^6 x + \frac{c_2^{12} - n^3(a c_2^6 - b_1^6)}{b_1^3} = \left(\frac{b_1^2 n + c_2^4}{b_1} \right)^3.$$

$$\therefore x = \frac{c^2 n}{b} \text{ renders } ax^3 + 3b^3 x^2 + 3c^6 x + \frac{c^{12} - n^3(a c^6 - b^6)}{b^3} = \left(\frac{b^2 n + c^4}{b} \right)^3 \quad \dots (D''')$$

Cor. Put $b^2 n + c^4 = 0$, or $n = -c^4 \div b^2$, in (D''') and reduce, then $x = -c^6 \div b^3$ renders $ax^3 + 3b^3 x^2 + 3c^6 x + a c^{18} \div b^9 = 0, \dots (d''')$